Error Estimates for Discrete Derivatives

Recall: Discretization of DE had following steps:

Differential Equation

$$y'' + p(t)y' + q(t)y = f(t)$$

① Evaluate at sample points t_k

System of Eqns in y_k'', y_k', y_k

$$\begin{bmatrix} y_1'' + p(t_1)y_1' + q(t_1)y_1 = f(t_1) \\ y_2'' + p(t_2)y_2' + q(t_2)y_2 = f(t_2) \\ \vdots \end{bmatrix}$$

② Convert to y_k & plug in boundary

Equations using only y_k

$$\begin{bmatrix} (y_2 - 2y_1 + \overset{=\cdots}{\cancel{y_0}})/h^2 + p_1 \cdot (y_2 - \overset{=\cdots}{\cancel{y_0}})/2h + \cdots \\ (y_3 - 2y_2 + y_1)/h^2 + p_2 \cdot (y_3 - y_1)/2h + \cdots \\ \vdots \end{bmatrix}$$

③ Convert to matrix

Matrix Equation

$$\begin{bmatrix} \text{— DE —} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} \text{BV} \end{bmatrix}$$

"y_0"

"y_{n+1}"

Step #2 used the following formulas estimating discretization of derivatives:

1st Order DE	2nd Order DE
$y_k' = (y_{k+1} - y_k)/h$	$y_k'' = (y_{k+1} - 2y_k + y_{k-1})/h^2$
— or —	
$y_k' = (y_k - y_{k-1})/h$	$y_k' = (y_{k+1} - y_{k-1})/2h$

Questions:

(1) How do we know these formulas are accurate?

(2) How accurate are these formulas? (How quickly do they converge?)

Error Estimates

Plan: Use Taylor series at sample point to estimate error size of discrete deriv.

Error Estimates (big theoretical computation)

- **Setup:** Taylor series for $y(t)$ at sample point t_k

$$y(t) = y(t_k) + y'(t_k)\cdot(t-t_k) + \frac{y''(t_k)}{2}(t-t_k)^2 + \frac{y'''(t_k)}{6}(t-t_k)^3 + \cdots$$

Plug in t_{k-1}, t_k, t_{k+1} to get series for y_{k-1}, y_k, y_{k+1} (Note: $t_{k+1} - t_k = h$)

$$y_{k+1} = y(t_k) + y'(t_k)\cdot h + \frac{y''(t_k)}{2}\cdot h^2 + \frac{y'''(t_k)}{6}\cdot h^3 + \cdots$$

$$y_k = y(t_k) + 0$$

$$y_{k-1} = y(t_k) + y'(t_k)(-h) + \frac{y''(t_k)}{2}\cdot(-h)^2 + \frac{y'''(t_k)}{6}(-h)^3 + \cdots$$

Plugging into "Forward Difference" gives

$$y_k' = (y_{k+1} - y_k)/h$$

$$= \left(y'(t_k)\cdot h + \frac{y''(t_k)}{2}\cdot h^2 + \frac{y'''(t_k)}{6} + \cdots\right)/h$$

$$= y'(t_k) + \underbrace{\frac{y''(t_k)}{2}\cdot h + \frac{y'''(t_k)}{6}\cdot h^2 + \cdots}_{\text{Error}}$$

When h is small, the dominant term in Error will be

$$\frac{y''(t_k)}{2}\cdot h$$

We say
- Error is proportional to y''
- Error decreases linearly with h

Plugging into "Backward Difference" gives essentially the same result

→ Now let's look at 2nd order formulas

Plugging into "Centered Difference" gives

$$y_k' = (y_{k+1} - y_{k-1})/2h$$

$$= \left[\begin{array}{ccccccccc}
\cancel{y(t_k)} & + & y'(t_k)\cdot h & + & \cancel{\dfrac{y''(t_k)}{2}}\cdot h^2 & + & \dfrac{y'''(t_k)}{6}\cdot h^3 & + \cdots & \\
-\big(\cancel{y(t_k)} & + & y'(t_k)(-h) & + & \cancel{\dfrac{y''(t_k)}{2}}(-h)^2 & + & \dfrac{y'''(t_k)}{6}(-h)^3 & + \cdots & \big) \\
\text{cancel} & & \text{add} & & \text{cancel} & & \text{add} & &
\end{array}\right] /2h$$

$$= \left[\; 2y'(t_k)\cdot h \;+\; \frac{y'''(t_k)}{3} h^3 \;+ \cdots \right] /2h$$

$$= y'(t_k) + \underbrace{\boxed{\frac{y'''(t_k)}{6} h^2 + \cdots}}_{\text{Error}}$$

Note: Error for "centered diff" decreases with h^2 !!!

$$h = \frac{b-a}{n+1}$$

So $\underline{\underline{10}}$ points of centered diff $\approx$ $\underline{\underline{100}}$ points of forward diff

Plugging into "Second Difference" gives

$$y_k'' = (y_{k+1} - 2y_k + y_{k-1})/h^2$$

$$= \left[\begin{array}{ccccccccccc}
\cancel{y(t_k)} & + & \cancel{y'(t_k)}h & + & \dfrac{y''(t_k)}{2}h^2 & + & \cancel{\dfrac{y'''(t_k)}{6}}h^3 & + & \dfrac{y''''(t_k)}{24}h^4 & + \cdots \\
-\cancel{2y(t_k)} & & & & & & & & & \\
+\cancel{y(t_k)} & + & \cancel{y'(t_k)}(-h) & + & \dfrac{y''(t_k)}{2}(-h)^2 & + & \cancel{\dfrac{y'''(t_k)}{6}}(-h)^3 & + & \dfrac{y''''(t_k)}{24}h^4 & + \cdots \\
\text{cancel} & & \text{cancel} & & \text{add} & & \text{cancel} & & \text{add} &
\end{array}\right] /h^2$$

(continuing computation)

$$y_k'' = \cdots$$

$$= \left[y''(t_k)\,h^2 + \frac{y''''(t_k)}{12}\,h^4 + \cdots \right] / h^2$$

$$= y''(t_k) + \boxed{\frac{y''''(t_k)}{12}\,h^2 + \cdots}$$

Error

$\Rightarrow$

Note: Error for "Second Diff" also decreases with h^2 !!

So 2nd Order DE discretization has Error proportional to h^2.

Similar computations can be used to verify "higher difference formulas":

(Forward) Third Diff $\quad y_k''' = y_{k+2} - 3y_{k+1} + 3y_k - y_{k-1}$ $\quad$ 3rd Order DE (Error $\approx h^3$)

Fourth Diff $\quad y_k'''' = y_{k+2} - 4y_{k+1} + 6y_k - 4y_{k-1} + y_{k-2}$

(Centered) Third Diff $\quad y_k''' = -y_{k+2} + 2y_{k+1} - 2y_{k-1} + y_{k-2}$

4th Order DE (Error $\approx h^4$)

etc.